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The ranks of partitions modulo 2

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Abstract

Let $N(0, 2, n)$, respectively $N(1, 2, n)$, denote the number of partitions of n whose ranks are even, respectively odd. We show here that $N(0, 2, n) < N(1, 2, n)$, when n is even, and that this inequality is reversed, when n is odd. Our proof is 'bijective' in that we construct an injective map between the sets of partitions involved. We use a variation of the Involution Principle of Garsia and Milne.

0. Introduction

A partition π is a finite multiset of positive integers (the *parts* of π). We denote the largest part of π by π_0 and the number of parts by $\#\pi$. The *weight* of π, denoted $\omega(\pi)$, is the sum of the parts of π. If $\omega(\pi) = n$, π is a *partition of n*.

In 1994, Freeman Dyson [2] defined the *rank* of a partition π by

$$\mathrm{rank}(\pi) := \pi_0 - \#\pi.$$

Setting

$$N(r, m, n) := \#\{\pi\colon \omega(\pi) = n,\ \mathrm{rank}(\pi) \equiv r \bmod m\}$$

he found empirically that

$$N(0, 5, 5n + 4) = N(1, 5, 5n + 4) = N(2, 5, 5n + 4) \tag{1}$$

and

$$N(0, 7, 7n + 5) = N(1, 7, 7n + 5) = N(2, 7, 7n + 5) = N(3, 7, 7n + 5). \tag{2}$$

(He also noticed several other relations between the numbers $N(r, 5, n)$ and between the numbers $N(r, 7, n)$). Now it is easy to see that $N(r, m, n) = N(-r, m, n)$ and

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so Dyson's observations provide concrete realisations of the Ramanujan congruences [8]

$$p(5n+4) \equiv 0 \bmod 5 \quad \text{and} \quad p(7n+5) \equiv 0 \bmod 7.$$

However, as Dyson observed, the rank does not serve to realise the third Ramanujan congruence, $p(11n+6) \equiv 0 \bmod 11$ in the same way.

All Dyson's observations were proved by Atkin and Swinnerton-Dyer in 1954 [1]. They used heavy analytical methods in their proof and, since (1) and (2) are purely combinatorial statements, it would be satisfactory if there were combinatorial (bijective) proofs of these statements. Such proofs are not yet available (though Garvan et al. [5] have given combinatorial proofs of the Ramanujan congruences); indeed, apart from Dyson's own derivation of the generating function of the numbers $N(m, n)$ [3], I know of no combinatorial treatment of any result concerning Dyson's rank. My modest purpose here is to give a bijective proof of

Theorem

$$N(0, 2, 2n) < N(1, 2, 2n) \quad \textit{if } n > 0,$$

$$N(0, 2, 2n+1) > N(1, 2, 2n+1) \quad \textit{if } n \geqslant 0.$$

Set

$$\boldsymbol{R}(\varepsilon) = \{\text{partitions } \pi\text{: } \omega(\pi) > 0,\ \omega(\pi) + \pi_0 + \#\pi \equiv \varepsilon \bmod 2\}.$$

Then we prove the theorem (with weak inequalities) by constructing an injective map

$$\Phi: \boldsymbol{R}(0) \to \boldsymbol{R}(1).$$

The theorem itself is then established by noting that none of the partitions $1 + 1 + \cdots + 1$, which lie in $\boldsymbol{R}(1)$, lies in the image of Φ.

1. The involution principle

The Involution Principle was introduced by Garsia and Milne [4] in their bijective proof of the Rogers–Ramanujan identities. We give a summary of the technique.

Suppose X is a set each of whose elements has *weight* $\omega(x)$, a nonnegative integer, and also that each element of X has a *sign*, $\operatorname{sign}(x) = +$ or $-$. Such sets are called WS-sets in [7]. If X and Y are WS-sets, we regard $X \times Y$ as a WS-set with

$$\omega(x, y) = \omega(x) + \omega(y) \quad \text{and} \quad \operatorname{sign}(x, y) = \operatorname{sign}(x)\operatorname{sign}(y).$$

We say that α is a *wsp* (for *weight-preserving, sign-changing, partial*) *involution* on X if there is a subset $X^\alpha \subseteq X$ such that $\alpha: X^\alpha \to X^\alpha$ is an involution with the property:

$$\omega(\alpha x) = \omega(x) \quad \text{and} \quad \operatorname{sign}(\alpha x) = -\operatorname{sign}(x)$$

for each $x \in X^\alpha$. Set $X_\alpha = X - X^\alpha$, i.e. the subset of X on which α is not defined. We shall refer to the set X_α as the *fixed-point set* of α. Define a *GM-pair* to be a pair (α, β) of wsp-involutions on a WS-set X such that $X^{\alpha} \cap X^{\beta}$ is locally finite, meaning that each subset $\{x\colon \omega(x) = n\}$ is finite.

Suppose (α, β) is a GM-pair on X. It is evident from the finiteness condition that, for each $x \in X_{\alpha}$, the sequence

$$x, \beta x, \alpha\beta x, \beta\alpha\beta x, \ldots$$

eventually comes to rest at some point

$$\boldsymbol{gm}_{\alpha}(x) = \beta^{\varepsilon}(\alpha\beta)^{n}(x) \quad \text{with } \varepsilon = 0 \text{ or } 1,\ n \geqslant 0.$$

$$\in \begin{cases} X_{\beta} & \text{if } \varepsilon = 0, \\ X_{\alpha} & \text{if } \varepsilon = 1. \end{cases}$$

(Note that, if $x \in X_{\alpha} \cap X_{\beta}$, the sequence above just contains the term x and so $\boldsymbol{gm}_{\alpha}(x) = x$). Since α and β are sign-changing we have,

$$x \in X_{\alpha}^{+/-} \Rightarrow \boldsymbol{gm}_{\alpha}(x) \in X_{\alpha}^{-/+} \cup X_{\beta}^{+/-}.$$

Finally, it is obvious that $\boldsymbol{gm}_{\alpha}$ preserves weights and a little thought shows that

$$\boldsymbol{gm}_{\alpha} \text{ is injective.} \tag{3}$$

(This is because the map $\boldsymbol{gm}$, defined on $X_{\alpha} \cup X_{\beta}$ to be $\boldsymbol{gm}_{\alpha}$ on X_{α} and $\boldsymbol{gm}_{\beta}$ on X_{β}, is an involution.)

2. The construction

If γ is any set of partitions, we regard γ as a WS-set, taking the weight of $\pi \in \gamma$ to be the sum of its parts and with $\operatorname{sign}(\pi) = (-)^{\omega(\pi)}$. We also define another WS-set $\sigma\gamma$, with underlying set γ and the same weights but with $\operatorname{sign}(\pi) = (-)^{\omega(\pi) + \#\pi}$

If W is a set of positive integers, set

$$\boldsymbol{E}(W) := \{\text{partitions } \pi\colon \text{each part lies in } W\}$$

$$\boldsymbol{E}_{\mathrm{d}}(W) := \{\pi \in \boldsymbol{E}(W)\colon \pi \text{ has distinct parts}\}$$

The *cancelling involution* [4], γ, is defined on a pair $(\pi, \rho) \in \boldsymbol{E}(W) \times \sigma\boldsymbol{E}_{\mathrm{d}}(W)$, where π and ρ are not both empty, by

$$\gamma(\pi, \rho) = \begin{cases} (\pi - \{\pi_0\}, \rho \cup \{\pi_0\}) & \text{if } \pi_0 > \rho_0, \\ (\pi \cup \{\rho_0\}, \rho - \{\rho_0\}) & \text{if } \pi_0 \leqslant \rho_0 \end{cases}$$

It is plain that γ is weight-preserving and sign-changing and has fixed-point set $\{(\emptyset, \emptyset)\}$. The same construction gives a wsp-involution on $\sigma\boldsymbol{E}(W) \times \boldsymbol{E}_{\mathrm{d}}(W)$, which we also call a cancelling involution.

Define the sets

$$\boldsymbol{E}(m) := \boldsymbol{E}(\{1, \ldots, m\}) = \{\text{partitions } \pi\colon \pi_0 \leqslant m\}$$

$$\boldsymbol{O}(m) := \{\text{partitions } \pi\colon \text{each } \pi_i \text{ odd and } \leqslant m\}$$

$$\boldsymbol{B}(m) := \{\text{partitions } \pi\colon \text{each } \pi_i \text{ even and } m < \pi_i \leqslant 2m\}$$

and let $\boldsymbol{E}_{\mathrm{d}}(m)$, $\boldsymbol{O}_{\mathrm{d}}(m)$ and $\boldsymbol{B}_{\mathrm{d}}(m)$ denote the same sets, but with distinct parts. There is a weight-preserving bijection

$$\phi : \boldsymbol{E}_{\mathrm{d}}(m) \times \boldsymbol{B}(m) \to \boldsymbol{O}(m)$$

defined on a pair (π, ρ) by splitting the even parts of π and of ρ equally in two until only odd numbers remain. These numbers constitute $\phi(\pi, \rho)$. The inverse of ϕ is defined on $\sigma \in \boldsymbol{O}(m)$ by adding equal parts of σ over and over again until (if ever) a number greater than m appears. This number is thrown into the right-hand component of $\phi^{-1}\sigma$ and the process is repeated on the rest of σ. When this process can no longer be performed, we have distinct numbers, each $\leqslant m$, remaining in σ and these make up the left-hand component of $\phi^{-1}\sigma$. This bijection is essentially due to Glaisher [6].

Define the WS-set

$$\boldsymbol{X}(m) := \sigma\boldsymbol{E}(m) \times \boldsymbol{O}(m) \times \sigma\boldsymbol{B}_{\mathrm{d}}(m) \times \sigma\boldsymbol{O}_{\mathrm{d}}(m) \times \boldsymbol{B}(m)$$

and define a GM-pair (α, β) of wsp-involutions on $\boldsymbol{X}(m)$ in the following way. α has fixed-point set

$$\boldsymbol{X}(m)_\alpha = \sigma\boldsymbol{E}(m) \times \{\emptyset\} \times \{\emptyset\} \times \{\emptyset\} \times \{\emptyset\}$$

and is defined on $(\pi, \rho, \sigma, \tau, v) \in \boldsymbol{X}(m)^\alpha$ by the cancelling involution on (ρ, τ), if $(\rho, \tau) \neq (\emptyset, \emptyset)$ (leaving π, σ and v alone) and otherwise by doing the cancelling involution on (σ, v). β has fixed-point set

$$\boldsymbol{X}(m)_\beta = \{\emptyset\} \times \{\emptyset\} \times \{\emptyset\} \times \sigma\boldsymbol{O}_{\mathrm{d}}(m) \times \boldsymbol{B}(m)$$

and is defined on $(\pi, \rho, \sigma, \tau, v) \in \boldsymbol{X}(m)^\beta$ by the action of ϕ^{-1} on the second component to give $(\pi, \rho', \rho'', \sigma, \tau, v)$, followed by the cancelling involution on (ρ'', σ) if $(\rho'', \sigma) \neq (\emptyset, \emptyset)$, otherwise by the cancelling involution on (π, ρ'), and then acting on the second and third components by ϕ.

Now $\boldsymbol{X}(m)_\beta^-$ is plainly empty and, identifying $\boldsymbol{X}(m^-)_\alpha$ with $\sigma\boldsymbol{E}(m)^-$, it follows from (3) that $\boldsymbol{gm}_\alpha$ induces an injective map

$$f_m : \sigma\boldsymbol{E}(m)^- \to \sigma\boldsymbol{E}(m)^+.$$

Finally, we define the map $\Phi : \boldsymbol{R}(0) \to \boldsymbol{R}(1)$. If $\pi \in \boldsymbol{R}(0)$, then

$$\Phi(\pi) := \{\pi_0\} \cup \boldsymbol{f}_{\pi_0}(\pi - \{\pi_0\}).$$

It is not difficult to see that none of the partitions $111\ldots1 \in \boldsymbol{R}(1)$ lies in the image of Φ. Here is an example. Take the partition $6442 \in \boldsymbol{R}(0)$. Then $\Phi(6442) = \{6\} \cup \boldsymbol{f}_6(442)$ and the α–β sequence that defines $\boldsymbol{f}_6(442) = \boldsymbol{gm}_\alpha(442)$ proceeds thus:

$$(442,\emptyset,\emptyset,\emptyset,\emptyset) \xrightarrow{\beta} (42,1111,\emptyset,\emptyset,\emptyset) \xrightarrow{\alpha} (42,111,\emptyset,1,\emptyset) \xrightarrow{\beta} (2,111111,\emptyset,1,\emptyset)$$

$$\xrightarrow{\alpha} (2,11111111,\emptyset,\emptyset,\emptyset) \xrightarrow{\beta} (2,\emptyset,8,\emptyset,\emptyset) \xrightarrow{\alpha} (2,\emptyset,\emptyset,\emptyset,8) \xrightarrow{\beta} (\emptyset,11,\emptyset,\emptyset,8) \xrightarrow{\alpha} (\emptyset,1,\emptyset,1,8)$$

$$\xrightarrow{\beta} (1,\emptyset,\emptyset,1,8) \xrightarrow{\alpha} (1,1,\emptyset,\emptyset,8) \xrightarrow{\beta} (11,\emptyset,\emptyset,\emptyset,8) \xrightarrow{\alpha} (11,\emptyset,8,\emptyset,\emptyset) \xrightarrow{\beta} (11,11111111,\emptyset,\emptyset,\emptyset)$$

$$\xrightarrow{\alpha} (11,1111111,\emptyset,1,\emptyset) \xrightarrow{\beta} (411,111,\emptyset,1,\emptyset) \xrightarrow{\alpha} (411,1111,\emptyset,\emptyset,\emptyset) \xrightarrow{\beta} (4411,\emptyset,\emptyset,\emptyset,\emptyset)$$

The sequence comes to rest at this point and $\boldsymbol{f}_6(442) = 4411$. So $\Phi(6442) = 64411$.

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